MATHEMATICAL THEORY OF CONFOUNDING IN ASYMMETRICAL AND SYMMETRICAL FACTORIAL DESIGNS

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1. Introduction

The problem of confounding in the general symmetrical factorial design s^m, where s is a prime positive integer or a power of prime and m any positive integer, was solved by Bose and Kishen (1940) by representing each treatment combination by a finite point of the associated m-dimensional finite projective geometry $PG(m, s)$ constructed from the Galois field $GF(s)$ and using linear spaces or flats represented by linear equations in m variables. This method is not applicable in the construction of confounded symmetrical factorial designs s^m, where s is not a prime number or its power, nor in obtaining confounded designs in the general asymmetrical factorial experiment $s_1 \times s_2 \times \ldots \times s_m$, where $s_1, s_2, \ldots, s_m$ are not all equal. Special methods have, therefore, to be applied for the construction of such designs.

The problem of confounding in designs of the type $3^{m_1} \times 2^{m_2}$, where m_1, m_2 are any positive integers, and all cases reducible to it, has been completely solved by Yates (1937). Using methods similar to Yates's, Li (1944) has constructed confounded designs for the asymmetrical factorial experiments 4×2^2, 5×2^2, $4 \times 3 \times 2$, $4^2 \times 2$, 4×3^2, $4^2 \times 3$ and $4^2 \times 2$. Nair and Rao (1941, 1942) have developed a set of sufficient combinatorial conditions which lead to the construction of confounded designs of the general asymmetrical factorial experiment. Thompson and Dick (1951), starting from a basic $p \times q$ design in blocks of q plots ($q < p$, p being a prime number or a power of a prime), have obtained three-factor designs with the same block size, the number of levels being p, q or factors of q. Kishen (1958) has given balanced designs of the type $q \times 2^2$ and $q \times p^2$.

The method of finite geometries has been recently extended by Kishen and Srivastava (1959) to the construction of balanced confounded asymmetrical factorial designs. This has been done by using curvilinear spaces or hypersurfaces and truncating the *EG* (m, s) suitably. This method has been further developed in this paper and has been supplemented by more general methods using vectors in Galois fields. With the help of these methods, almost all confounded asymmetrical and symmetrical factorial designs having optimum properties have been constructed. The method of analysis of these designs has also been briefly discussed. The appropriateness of the large number of factorial designs that have now become available under experimental situations commonly encountered will be discussed in a separate communication.

2. Hypersurfaces in Finite Geometries

2.1. *Simple Hypersurfaces*

A hypersurface in *EG* (m, s) may be represented by the equation

$$\phi(x_1, x_2, \ldots, x_m) = 0 \qquad (1)$$

of which a particular case is given by the equation

$$a_0 + a_1 f_1(x_1) + a_2 f_2(x_2) + \ldots + a_m f_m(x_m) = 0 \qquad (2)$$

in which all the variables occur separately,

$a_0, a_1, a_2, \ldots, a_m$ being any elements of $GF(s)$

and

$$f_i(x) = a_{i1} x + a_{i2} x^2 + \ldots + a_{i,\, s-1} x^{s-1} \qquad (3)$$

where the a_{ij}'s are also elements of $EG(s)$.

When $f_i(x) = x^{b_i}$, we get simple hypersurfaces of the type

$$a_0 + a_1 x_1^{b_1} + a_2 x_2^{b_2} + \ldots + a_m x_m^{b_m} = 0 \quad (4)$$

which we shall consider first.

The following theorem will be proved in this connection:

THEOREM 2.1.—*If* d *is a divisor of* $s - 1 = p^n - 1$, *then* x^d *and* x^{s-1-d} *will give exactly* $(s-1)/d + 1$ *distinct values when* x *is varied from* α_0 *to* α_{s-1} *in* GF (s).

Let d be a divisor of $p^n - 1$. It is known that the equation $x^d = 1$ will have exactly d roots. Let these roots be $\mu_1, \mu_2, \ldots \mu_d$. Since $d < (s-1)$, there will be other elements not included in this set of μ's. Let ν_1 be such an element and let $\nu_1 d = \beta_1$. Then the equation $x^d = \beta_1$ will be satisfied by $\nu_1\mu_1, \nu_1\mu_2, \ldots, \nu_1\mu_d$. Let ν_2 be another element not included in the two sets (μ_i) and $(\nu_1\mu_i)$, and let $\nu_2^d = \beta_2$. Then the roots of $x^d = \beta_2$ will be $\nu_2\mu_1, \nu_2\mu_2, \ldots, \nu_2\mu_d$. If $(s-1) = qd$, then obviously we will get the sets $(\nu_3\mu_i), (\nu_4\mu_i), \ldots, (\nu_{q-1}\mu_i)$, and all these sets together will exhaust the $(s-1)$ elements (excluding zero) of $GF(s)$. The set $(\nu_r\mu_i)$ $(r = 0, 1, \ldots, q-1;\ \nu_0 = 1)$ will satisfy the equation $x^d = \beta_r$ (where $\beta_0 = 1$). Hence x^d will give q distinct values when x is varied from α_1 to α_{s-1} and these will be $\beta_0, \beta_1, \ldots, \beta_{q-1}$. Including $x = 0$, we shall thus obtain $(q+1)$ distinct values when $x = \alpha_0, \alpha_1, \ldots, \alpha_{s-1}$. Further, we know that all the elements of $GF(s)$ satisfy the equation $x^{p^n-1} = 1$. Hence $x^{s-1} = 0$ when $x = \alpha_0$ and equal to α_1 for all other values of x. Since, for $x = \alpha_1, \alpha_2, \ldots, \alpha_{s-1}$, we get q distinct values for x^d, we shall obtain q distinct values for $1/x^d$ and, consequently, also for x^{s-1}/x^d or x^{s-1-d}. Hence the theorem.

2.2. *Polynomials Yielding* k *Distinct Levels*

The question now arises whether it is possible to get k distinct levels by taking instead of $f_i(x) = x^{b_i}$ in equation (2), an appropriate polynomial in x, say,

$$y = f(x) = a_1 x + a_2 x^2 + \ldots + a_{s-1} x^{s-1} \quad (5)$$

where k is *any* number less than s. This means that $f(x)$ should be such that for $x = \alpha_0, \alpha_1, \ldots, \alpha_{s-1}, f(x)$ provides only k distinct values, say, $y_1, y_2, \ldots, y_k$. This result will be proved in the two theorems that follow.

THEOREM 2.2.—*The power-matrix*

$$S = \begin{bmatrix} a_1 & a_1^2 & \dots & a_1^t & \dots & a_1^{s-1} \\ a_2 & a_2^2 & \dots & a_2^t & \dots & a_2^{s-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_r & a_r^2 & \dots & a_r^t & \dots & a_r^{s-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{s-1} & a_{s-1}^2 & \dots & a_{s-1}^t & \dots & a_{s-1}^{s-1} \end{bmatrix} = [a_r^t] \qquad (6)$$

where a_r^t (r, t = 0, 1, ..., s − 1) *are all non-zero elements of* GF (s), *is of rank* (s − 1) *and its inverse is given by*

$$T = S^{-1} = a^{-1}{}_{p-1} \times \begin{bmatrix} a_1^{s-2} & a_2^{s-2} & \dots & a_t^{s-2} & \dots & a_{s-1}^{s-2} \\ a_1^{s-3} & a_2^{s-3} & \dots & a_t^{s-3} & \dots & a_{s-1}^{s-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1^{s-r-1} & a_2^{s-r-1} & \dots & a_t^{s-r-1} & \dots & a_{s-1}^{s-r-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & a_t & \dots & a_{s-1} \\ a_1 & a_1 & \dots & a_1 & \dots & a_1 \end{bmatrix} \qquad (7)$$

where $s = p^n$ $(n \geqslant 1)$.

Let us consider the product *ST*. The element in the *r*-th row and *t*-th column of (*ST*), where $r \neq t$, is given by the sum of products of elements in the *r*-th row of *S* and *t*-th column of *T*, and equals

$$a^{-1}{}_{p-1} [a_r a_t^{s-2} + a_r^2 a_t^{s-3} + \dots + a_r^i a_t^{s-i-1} + \dots + a_r^{s-1} a_t^0]$$

$$= a^{-1}{}_{p-1} a_t^{s-1} \left[\left(\frac{a_r}{a_t}\right) + \left(\frac{a_r}{a_t}\right)^2 + \dots + \left(\frac{a_r}{a_t}\right)^i \right.$$

$$\left. + \dots + \left(\frac{a_r}{a_t}\right)^{s-1} \right]$$

Now, since α_r and α_t are both non-zero elements of $GF(s)$, the quotient (α_r/α_t) exists. Let

$$\frac{\alpha_r}{\alpha_t} = \omega \neq 1 \tag{8}$$

The above expression then reduces to

$$\alpha^{-1}_{p-1}\alpha_t^{s-1}[\omega + \omega^2 + \omega^3 + \ldots + \omega^i \ldots + \omega^{s-1}] = 0$$

since

$$\omega \neq \alpha_1 \quad \text{and} \quad \omega^{s-1} = \alpha_1 \text{ for all } \omega\ (\neq 0).$$

Also, for $r = t$ the product (8) becomes

$$\alpha^{-1}_{p-1}\alpha_t^{s-1}[\alpha_1 + \alpha_1 + \ldots + \alpha_1 + \ldots (p-1) \text{ times}]$$
$$= \alpha^{-1}_{p-1} \times \alpha_1 \times \alpha_{p-1} = \alpha_1.$$

Hence the product ST is a unit matrix. Obviously, the rank of both S and T is $(s-1)$.

THEOREM 2.3.—

Let y and $f(x)$ be defined as in (5), and let

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \vdots \\ \cdot \\ a_{s-1} \end{bmatrix}. \quad \text{Here } f(0) = 0.$$

Then there exist a set of matrices such that as x is varied from α_1 to α_{s-1}, only $(k-1)$ distinct values of y other than α_0 are obtained, so that including $x = 0$ we have k distinct levels. Further, there will be a subset of this set of matrices such that the $(k-1)$ distinct values of y correspond to certain given values of x, say $x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}}$.

The proof is simple. Consider the product SA and let $Y = SA$. Obviously, since S is of rank $(s-1)$, Y exists and equals

$$Y = \begin{bmatrix} a_1\times\alpha_1 + a_2\times\alpha_1^2 + \ldots + a_{s-1}\times\alpha_1^{s-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_1\times\alpha_r + a_2\times\alpha_r^2 + \ldots + a_{s-1}\times\alpha_r^{s-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_1\times\alpha_{s-1} + a_2\times\alpha^2_{s-1} + \ldots + a_{s-1}\times\alpha_{s-1}^{s-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \vdots \\ \cdot \\ y_{s-1} \end{bmatrix}, \text{ say} \tag{9}$$

where $y_i = f(x_i)$. Now we want only $(k-1)$ distinct elements in the last matrix in (9). This may be done in $[(s-1)!/(k-1)!\ (s-k)!] \times k^{s-k}$ ways, corresponding to each of which there will exist a surface giving k distinct values. The polynomials yielding distinct fixed values y_{i_1}, $y_{i_2}, \ldots, y_{i_{k-1}}$ against fixed levels of $x = x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}}$ can be obtained in $(k)^{s-k}$ ways. The corresponding set of surfaces may be called the kth level isomorphic set of surfaces, and the corresponding set of A matrices will be given by $A = TY$, where the $(k-1)$ elements in Y are fixed and the rest may vary from α_0 to α_{s-1}.

3. ASYMMETRICAL CONFOUNDED DESIGNS

Let us now consider equation (4). If A_i is a factor which is included in equation (4) as $x_i^{b_i}$, the contribution made by it in the equation can take only $s_i = (s-1)/b_i + 1$ distinct values, s_i being thus less than s, assuming that b_i is a divisor of $(s-1)$. Let the s_i distinct values of $x_i^{b_i}$ correspond *in order* to the values of x_i equal to $\alpha_0, \alpha_1, \alpha_{i_1}, \alpha_{i_2}, \ldots,$ α_{i_s-2}. This means in effect that equation (4) will behave as if the j-th factor had only s_i distinct levels, namely,

$$(0, 1, j_1, j_2, \ldots, j_{s_i-2}) \tag{10}$$

In the context of asymmetrical designs, this suggests that the levels of x_i other than those given by (10) be left out of consideration and the Euclidean Geometry $EG(m, s)$ containing s^m points be so truncated that all the treatment combinations in which the above levels of A_i occur are cut out. Such a truncation may be done with respect to any number of factors, as required.

Consider now m factors $A_1, A_2, \ldots, A_m$ at levels $s_1, s_2, \ldots, s_m$ respectively, where s_1 is a prime number and $s_i \leqslant s_1$ for all $i > 1$. As shown in Section (2.2), it is possible to have s_i equal to any number less than s_1 by taking a suitable polynomial of x in $GF(s_1)$. Here for simplicity, we shall consider the case where the factors $A_2, A_3, \ldots, A_m$ corresponds to $x_2^{b_2}, x_3^{b_3}, \ldots, x_m^{b_m}$ respectively. Suppose now we desire to confound an m-factor interaction. We then take the pencil of hypersurfaces represented by

$$x_1 + [\alpha_{l_2} x_2^{t_2} + \alpha_{l_3} x_3^{t_3} + \ldots + \alpha_{l_m} x_m^{t_m}] = \alpha_r, \tag{11}$$

where $r = 0, 1, \ldots, (s-1)$, in the suitably truncated $EG(m, s_1)$, it being presumed that x_1 varies from α_0 to α_{s-1} and x_i $(i \neq 1)$ varies over $\alpha_0, \alpha_1, \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_{s_i}-2}$. We may now proceed to divide the $s_1 \times s_2 \times \ldots \times s_m$ treatment combinations in s_1 blocks of $s_2 \times s_3 \times \ldots \times s_m$ plots each with the help of the pencil (11). It can be shown that the pencil (11) will divide the treatment combinations symmetrically into s_1 sets. For, if $x_{2t_2}, x_{3t_3}, \ldots, x_{jt_j}, \ldots, x_{mt_m}$ is any combination of the levels of the factors $A_2, A_3, \ldots, A_j, \ldots, A_m$, the expression within brackets on the L.H.S. of equation (11) will have a fixed value, say, $A(t_2, \ldots, t_j, \ldots, t_m)$ in $GF(s_1)$. If $x_{1A\alpha}$ is the value of x_1 such that $x_{1A\alpha} + A = \alpha_r$ $(r = 0, 1, 2, \ldots, s-1)$, then the treatment combination $(x_{1A\alpha}, x_{2t_2}, \ldots, x_{mt_m})$ will appear in the α-th block. Thus, all the combinations of the levels of $x_2, x_3, \ldots, x_m$ will appear with different levels of x_1 in different blocks. Since x_1 can have s_1 values, we shall get s_1 blocks of equal size, each block containing all the $s_2 \times s_3 \times \ldots \times s_m$ combinations of $A_2, A_3, \ldots, A_m$.

It appears that in the replication provided by a pencil of hypersurfaces, the interactions confounded may belong to two types. The interaction corresponding to equation (11) which generates the replication is always partially confounded. This is, so to say, the deliberately confounded interaction. However, some of the interactions may get partially confounded automatically owing to the fact that the number of combinations of levels of factors to which they relate is not equal to, or a factor of, the block size. For example, in the $4 \times 2 \times 2$ design in blocks of 4 plots, our pencil will partially confound the *ABC* interaction in a particular replication, and the *AB* and *AC* interactions will also be partially confounded since there are 8 combinations of levels of *AB* and *AC* and the block size is only 4. Thus, in the replication corresponding to equation (11), the main effect *A* and all the interactions in which it enters will be partially confounded if $s_1 > s_i$ $(i = 2, \ldots, m)$.

For obtaining a design balanced with respect to all main effects and interactions, we may have to take all the replications obtained by varying the α_{l_j} $(j = 2, 3, \ldots, m)$ over $\alpha_1, \alpha_2, \ldots, \alpha_{s-1}$. Varying only a particular α_{l_j} (j fixed) from α_1 to α_{s-1} will mean, in a sense, balance over a particular contrast of all factors other than A_j.

When there are at least two factors at s_1 levels each, no main effect will be partially confounded. The interaction A_1A_2 will be partially confounded if only there is no third factor at s_1 levels, and so on. In the former case, varying l_2 in α_{l_2} from 1 to $(s_1 - 1)$ we shall obtain the $(s_1 - 1)$ replications required for balancing the A_1A_2 interaction with respect to the rest of the factors.

4. Illustrative Examples

4.1. *The* $3 \times 3 \times 2$ *Design in Blocks of* 6 *Plots*

Let the three factors be $A(0, 1, 2)$, $B(0, 1, 2)$ and $C(0, 1)$. In $EG(3, 3)$, we truncate all the points with $x_3 = 2$, since $x^2 = 0, 1, 1$ for

$x = 0, 1, 2$ and $x = 2$ does not give a distinct value for x^2. The truncated geometry will have 18 points left corresponding to the 18 treatment combinations. The balanced design in two replications, confounding *AB* (*J*) and *ABC* (*J*), is generated by the pencils of hypersurfaces

$$\left.\begin{array}{l} x_1 + x_2 + \ x_3{}^2 = 0, 1, 2 \\ x_1 + x_2 + 2x_3{}^2 = 0, 1, 2 \end{array}\right\} \qquad (12)$$

For obtaining complete balance on *AB*, we may take two more replications generated by the two pencils

$$\left.\begin{array}{l} x_1 + 2x_2 + x_3{}^2 \ = 0, 1, 2 \\ x_1 + 2x_2 + 2x_3{}^2 = 0, 1, 2 \end{array}\right\} \qquad (13)$$

which, by themselves, provide a balanced design partially confounding *AB* (*I*) and *ABC* (*I*).

The above can be easily generalized to obtain $3^{n-1} \times 2$ confounded designs in blocks of $3^{n-2} \times 2$ plots.

4.2. *The* $s^2 \times q$ *Design in Blocks of* sq *Plots, Balanced in* (s − 1) *Replications; where* s > q

Let the q levels be obtained by taking the polynomial $f(x)$. Let the factors be $A(0, 1, \ldots, s-1)$, $B(0, 1, \ldots, s-1)$ and $C(0, 1, 2, \ldots, q-1)$. Let us consider the replication given by the pencil

$$x_1 + \alpha_{i_2} x_2 + \alpha_{i_3} f(x_3) = \alpha_r \; (r = 0, 1, \ldots, s-1; \; i_2, i_3 \text{ fixed}) \qquad (14)$$

which will partially confound the *AB* and *ABC* interactions. The $(s-1)$ replications obtained by allowing i_3 to vary from 1 to $(s-1)$ will give a balanced set. For complete balancing with respect to *AB*, i_2 also is to be varied from 1 to $(s-1)$, giving the full set of $(s-1)^2$ replications.

Some of the more useful designs derived from this series are $3^2 \times 2$, $4^2 \times 2$, $5^2 \times 2$, $7^2 \times 2$, $8^2 \times 2$, $4^2 \times 3$ and $5^2 \times 3$ in blocks of 6, 8, 10, 14, 16, 12 and 15 plots respectively. As an illustration, the design $5 \times 5 \times 3$ in 15 plot blocks is given in Table I, where the c_i's denote the level of the factor C, and X_i's denote the sets of *AB* :

X_0: $a_0b_0, a_1b_4, a_2b_3, a_3b_2, a_4b_1$

X_1: $a_0b_1, a_1b_0, a_2b_4, a_3b_3, a_4b_2$

X_2: $a_0b_2, a_1b_1, a_2b_0, a_3b_4, a_4b_3$

X_3: $a_0b_3, a_1b_2, a_2b_1, a_3b_0, a_4b_4$

X_4: $a_0b_4, a_1b_3, a_2b_2, a_3b_1, a_4b_0$

The relative loss of information in an $s^2 \times q$ design, on each of the $(s-1)$ confounded *d.f.* of AB is $(s-q)/q(s-1)$ and on each of the $(s-1)(q-1)$ confounded *d.f.* of ABC, it is $s/\{q(s-1)\}$.

TABLE I

$5 \times 5 \times 3$ *balanced design in* 15 *plot blocks involving four replications*

	Replication I					Replication II				
Block No.	1	2	3	4	5	6	7	8	9	10
	X_0c_0	X_1c_0	X_2c_0	X_3c_0	X_4c_0	X_0c_0	X_1c_0	X_2c_0	X_3c_0	X_4c_0
	X_4c_1	X_0c_1	X_1c_1	X_2c_1	X_3c_1	X_3c_1	X_4c_1	X_0c_1	X_1c_1	X_2c_1
	X_3c_2	X_4c_2	X_0c_2	X_1c_2	X_2c_2	X_1c_2	X_2c_2	X_3c_2	X_4c_2	X_0c_2
	Replication III					Replication IV				
Block No.	11	12	13	14	15	16	17	18	19	20
	X_0c_0	X_1c_0	X_2c_0	X_3c_0	X_4c_0	X_0c_0	X_1c_0	X_2c_0	X_3c_0	X_4c_0
	X_2c_1	X_3c_1	X_4c_1	X_0c_1	X_1c_1	X_1c_1	X_2c_1	X_3c_1	X_4c_1	X_0c_1
	X_4c_2	X_0c_2	X_1c_2	X_2c_2	X_3c_2	X_2c_2	X_3c_2	X_4c_2	X_0c_2	X_1c_2

The above can be easily generalized to the corresponding $s^{m-r} \times q^r$ designs in blocks of $s^{m-r-1} \times q^r$ plots, *e.g.*, $3 \times 3 \times 2 \times 2$ design in 12 plot blocks and $4 \times 4 \times 2 \times 2$ in 16 plot blocks.

4.3. $s \times q_1 \times q_2$ *Design in Blocks of* q_1q_2 *Plots*, $(s \geqslant q_1, q_2)$, *Balanced in* $(s-1)^2$ *Replications*

In this design, the main effect A is confounded partially.

Many useful designs, *e.g.*, $s \times 3 \times 2$, $s \times 4 \times 2$, $s \times 2 \times 2$, $s \times 4 \times 3$, $s \times 5 \times 2$, etc. in blocks of plots 6, 8, 4, 12 and 10 may be derived from this general design. If $s > q_1q_2$, balancing may be achieved in $(s-1)$ replications only.

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4.4. $4 \times 3 \times 2 \times 2$ *Design in Blocks of* 12 *Plots*

Consider $GF(2^2)$ with minimum function $x^2 = x + 1$ and elements α_0, $\alpha_1 = 1$, $\alpha_2 = x$, $\alpha_3 = x^2 = x + 1$. Denote the factors by $A(0, 1, 2, 3)$, $B(0, 1, 2)$, $C(0, 1)$, $D(0, 1)$. The functions $f(x)$ corresponding to C and D may be taken as $x_3{}^3$ and $x_4{}^3$ respectively. For B, let us choose $f(x) = \alpha_0, \alpha_1, \alpha_3$ and α_0 respectively corresponding to $x = \alpha_0, \alpha_1, \alpha_2$ and α_3. Then, from (9), we have

$$A = \alpha_1{}^{-1} \begin{bmatrix} \alpha_1 & \alpha_2{}^2 & \alpha_3{}^2 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_1 & \alpha_1 \end{bmatrix} \times \begin{bmatrix} \alpha_1 \\ \alpha_3 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_3{}^2 \\ \alpha_1 + \alpha_2\alpha_3 \\ \alpha_1 + \alpha_3 \end{bmatrix} = \begin{bmatrix} \alpha_3 \\ \alpha_0 \\ \alpha_2 \end{bmatrix} \quad (15)$$

Hence $f(x_2) = \alpha_3 x_2 + \alpha_2 x_2{}^3$.

The pencil which confounds $ABCD$ may be represented by the equation

$$x_1 + (\alpha_3 x_2 + \alpha_2 x_2{}^3) + [x_3{}^3 + \alpha_3 x_4{}^3] = \alpha_r \; (r = 0, 1, 2) \quad (16)$$

in the truncated geometry EG (4, 4). The replication generated also partially confounds the interaction AC, AD, ACD, ABC and ABD. A design in 3 replications providing balance over AC and AD is given in Table II.

TABLE II

$4 \times 3 \times 2 \times 2$ *Balanced design in* 12 *plot blocks, involving three replications*

		Level of C, D											
Replication	..	I				II				III			
Block No.	..	1	2	3	4	5	6	7	8	9	10	11	12
Level of A,B													
00	..	00	10	11	01	00	01	10	11	00	11	01	10
01	..	10	00	01	11	01	00	11	10	11	00	10	01
02	..	01	11	10	00	11	10	01	00	10	01	11	00
10	..	10	00	01	11	01	00	11	10	11	00	10	01
11	..	00	10	11	01	00	01	10	11	00	11	01	10
12	..	11	01	00	10	10	11	00	01	01	10	00	11
20	..	11	01	00	10	10	11	00	01	01	10	00	11
21	..	01	11	10	00	11	10	01	00	10	01	11	00
22	..	10	00	01	11	01	00	11	10	11	00	10	01
30	..	01	11	10	00	11	10	01	00	10	01	11	00
31	..	11	01	00	10	10	11	00	01	01	10	00	11
32	..	00	10	11	01	00	01	10	11	00	11	01	10

It has to be noted that the coefficients of x_3^3 and x_4^3 have been kept different in (16). This has to be done as otherwise one degree of freedom belonging to the interaction AB will also be totally confounded. The reason is that if their coefficients are not distinct, x_3^3 and x_4^3, when combined together, will not generate all the elements of $GF(2^2)$.

4.5. $s_1 \times s_2 \times s_3 \times \ldots \times s_m$ *Design where* s_i *are equal to, or are Powers of, a Prime Number* p

As a special case of this design, let us consider the $4\times2\times2$ design in blocks of 4 plots. Denote the factors by $A\,(0, 1, 2, 3)$, $B\,(0, 1)$ and $C\,(0, 1)$. A suitable pencil confounding the ABC interaction is represented by

$$x_1 + \alpha_{i_1}(x_2^3 + \alpha_2 x_3^3) = \alpha_r \; (r = 0, 1, 2, 3;\; i_1 \text{ fixed}) \tag{17}$$

This also partially confounds the AB and AC interactions. A design in 3 replications is obtained by taking $i_1 = 1, 2$ and 3 and is shown in Table III below, in which the confounded interactions are also given.

TABLE III

$4\times2\times2$ *Balanced design in* 4 *plot blocks*

Replication ..	I				II				III			
Block No. ..	1	2	3	4	1	2	3	4	1	2	3	4
	000	100	200	300	000	100	200	300	000	100	200	300
	110	010	310	210	111	011	311	211	101	001	301	201
	201	301	001	101	210	310	010	110	211	311	011	111
	311	211	111	011	301	201	101	001	310	210	110	010

Confounded Interactions	$A'C, A'''B$ $A''BC$	$A'''C, A''B$ $A'BC$	$A'B, A''C$ $A'''BC$

Here $A' = (a_3 + a_2 - a_1 - a_0)$, $A'' = (a_3 - a_2 - a_1 + a_0)$ and $A''' = (a_3 - a_2 + a_1 - a_0)$.

The loss of information on each of the AB, AC and ABC interactions is 1/3. The above design can be immediately extended to 4×2^n designs in blocks of 2^n plots, balanced in 3 replications, as above.

Designs of the type $9 \times 3 \times 3$ in blocks of 9 plots, $16 \times 4 \times 4$ in blocks of 16 plots, $8 \times 2 \times 2 \times 2$ in blocks of 8 plots, $8 \times 4 \times 2$ in blocks of 8 plots, etc., can also be constructed by similar methods. These are balanced in 8, 15, 7 and 7 replications respectively. These confounded designs are amenable to arrangement in quasi-Latin squares.

4.6. $s_1 \times s_2 \times s_3 \times \ldots \times s_m$ *Design in Blocks of* $s_1 \times s_3 \times s_4 \times \ldots \times s_m$ *plots where* s_2 *is a factor of* $s_1 \times s_3 \times s_4 \times \ldots \times s_m$ *and is a Prime Number or a Power of a Prime, and* $s_2^2 \geqslant s_i \; (i \neq 2)$

Consider $GF(s_2^2)$. By Theorem 2.2, we can obtain s_i distinct levels from a suitable polynomial in $GF(s_2^2)$. Then a pencil of hypersurfaces will divide the total number of treatment combinations into s_2^2 blocks, each block containing $(1/s_2)(s_1 \times s_3 \times s_4 \times \ldots \times s_m)$ plots. We may then suitably combine sets of s_2 blocks out of these s_2^2 blocks to get s_2 new blocks, each containing $s_1 \times s_3 \times \ldots \times s_m$ plots.

4.7. $s^3 \times s_4 \times s_5 \ldots \times s_m$ *Design in* s^2 *Blocks of* $s \times s_4 \times s_5 \times \ldots \times s_m$ *Plots each*

Here 3 factors have been taken at s levels so that the number of plots in each of the s^2 blocks may still remain a multiple of s so as to keep all the main effects unconfounded. As in the symmetrical case of (s^m, s^2), we have here to confound two pencils simultaneously.

As an illustration, let us take the $5^3 \times 2$ design in blocks of 10 plots each.

Let us take the pencils

$$\left.\begin{array}{l} x_1 + 2x_2 + 2x_4^4 = 0, 1, 2, 3, 4 \\ \text{and} \\ x_1 + x_3 + x_4^4 = 0, 1, 2, 3, 4 \end{array}\right\} \quad (18)$$

in the truncated $EG(4, 5)$. Balance on any particular contrast belonging to the first three factors A, B and C can be achieved in 4 replications.

A generalization of the above procedure leads to the construction of balanced confounded designs of the type $s^{m_1} \times s_1^{l_1} \times s_2^{l_2} \times \ldots \times s_p^{l_p}$ in s^k blocks of $s^{m_1-k} \times s_1^{l_1} \times s_2^{l_2} \times \ldots \times s_p^{l_p}$ plots each where $k < m_1$; balancing being achieved in $(s-1)$ replications only, if $k \leqslant (s-1)$.

4.8. *Method of Cutting out from an* s^m *Design*

Suppose we have got an s^m design in s^k blocks of s^{m-k} plots each, where s is a prime power. Then it can be easily seen that we can derive a design of the type $s^k \times s_{k+1} \times s_{k+2} \times \ldots \times s_m$ where $s_i \leqslant s$, from it by cutting out in all blocks all the treatment combinations which contain any out of the last $(s - s_i)$ levels of the factor A_i, i varying from $k + 1$ to m. This method is essentially equivalent to cutting out points lying on the $(m - 1)$-flats $x_i = \alpha_r$ [r varying from s_i to $(s - 1)$; $i = k + 1, \ldots, m$] from a set of pencils of linear $(m - 1)$-flats giving the confounded symmetrical design. The designs obtained by this method will, however, all correspond to simple confounding to be described in the next section, and are obviously a particular case of the designs obtainable from hypersurfaces. However, as would appear from the foregoing sections, the hypersurfaces provide a natural representation of all asymmetrical designs which are derivable by the above method of cutting.

5. Use of Galois Fields in Confounding in Factorial Designs

Let us now examine the role played by Galois fields and finite geometries in the construction of confounded factorial designs. In the case of confounding in symmetrical designs with s^m treatment combinations, the number s enters both as the level of each of the m factors and is also used in the pencils in finite geometries in splitting up the s^m treatment combinations symmetrically into s parts. We have seen, however, that with truncated geometries, the levels of each factor may not be the same and still the use of $GF(s)$ leads us to s symmetrical partitions. If the total number of treatment combinations is v and we want s blocks in which the treatments occur symmetrically, evidently s should firstly be a factor of v, which means that at least one factor is to be at s levels. As we have shown in Section 3, we can, in that case, put all the treatments v into a sort of correspondence with the s elements of $GF(s)$. The construction of a confounded factorial design necessarily involves the partitioning of treatments into s parts, *i.e.*, putting the v objects into correspondence with the s blocks. The $GF(s)$ is thus simply a mathematical device for effecting such a correspondence.

The above suggests that we may construct the (s^m, s^k) design directly from $GF(s^k)$. This procedure should appear to be more natural than the ordinary one inasmuch as the blocks required correspond one-to-one to the elements

$$\alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{s^k-2}, \alpha_{s^k-1} \text{ of } GF(s^k).$$

It also appears that the use of $GF(s)$ to group the factors even when all of them are not at s levels may also be possible and may lead us to interesting designs since the groupings made by $GF(s)$ are in a sense symmetrical.

5.1. *General Theory for Symmetrical Case*

For developing the above approach, the use of vectors in Galois fields will be made. We first define the basic terminology.

(*a*) Any set of n elements in $GF(s)$ will be called an n-vector in $GF(s)$.

(*b*) Corresponding to a factor A at k levels, the vector $(0, 1, 2, \ldots, k-1)$ in the real field will be called the *Level Vector of* A.

(*c*) Corresponding to the level vector of A, there is an Associated Vector $(\beta_0, \beta_1, \beta_2, \ldots, \beta_{k-1})$ of A, where the β's are all elements of $GF(s)$, not necessarily distinct.

(*d*) Any vector in $GF(s)$ used to generate the required design will be called a Generator. With m factors, the generator will be an m-vector in $GF(s)$.

(*e*) The *sum* S and *product* P of two vectors $(a_1, a_2, \ldots, a_m)$ and $(b_1, b_2, \ldots, b_m)$ in $GF(s)$ will be $S = (a_1 + b_1, a_2 + b_2, \ldots, a_m + b_m)$ and $P = (a_1b_1, a_2b_2, \ldots, a_mb_m)$ while their *product sum* will be $Q = a_1b_1 + a_2b_2 + \ldots + a_mb_m$.

(*f*) If the elements of the Associated Vector of a factor correspond one-to-one to the elements of the Level Vector, all the confounded interactions in which the factor enters may be said to be *simply confounded* where this is not the case the confounding is said to be non-simple.

With respect to a particular generator, the set of all treatment vectors, which we may call treatment space, may be divided into s parts, the jth part containing those treatment vectors the associated vectors corresponding to which give a product sum α_j when multiplied by the generator. Here α_j is the $(j+1)$-th element of $GF(s)$. Since the usual arrangement of the s elements of $GF(s)$ in the order $\alpha_0 = 0, \alpha_1 = 1$, $\alpha_2 = \theta, \ldots, \alpha_j = \theta^{j-1}, \ldots, \alpha_{S-1} = \theta^{s-2}$ presents difficulties in the addition of elements when $n \neq 1$ $(s = p^n)$, it will be convenient to have a rearranged form for the elements of $GF(s)$. In $GF(s = p^n)$, where p is a prime number, the minimum function is of order n and of the form

$$\theta^n = \mu_0 + \mu_1\theta + \mu_2\theta^2 + \ldots + \mu_{n-1}\theta^{n-1} \tag{24}$$

where μ_i are elements of $GF(p)$. Hence any element of $GF(p^n)$ may be represented in the form

$$g_0 + g_1\theta + g_2\theta^2 + \ldots + g_{n-1}\theta^{n-1} \tag{25}$$

where the g_i are elements of $GF(p)$. Further, the elements of $GF(p^n)$ will be so arranged that (25) is the $(1 + g_0 + g_1 p + g_2 p^2 + \ldots + g_{n-1} p^{n-1})$-th element, where the g's and p will be taken as belonging to the real field. Thus, in this rearranged form for the elements of $GF(3^2)$, $\theta + 2$ will be the $(1 + 2 + 1 \times 3)$-th, or the 6th element.

For simplicity, let us consider $(3^3, 3)$ design with blocks of 9 plots. Here we require 3 partitions. Hence we use $GF(3)$. The level vectors are given by $A(0, 1, 2)$, $B(0, 1, 2)$ and $C(0, 1, 2)$. Let us have simple confounding so that the effect vectors are $(0, 1, 2)$ or $(0, 2, 1)$. Now consider the different forms of generators. A generator like $(1, 0, 0)$ will divide the treatment space into 3 parts, the j-th part containing all the treatment vectors containing the level a_j of A, which implies that the main effect is confounded. Considering the generator $(1, 1, 0)$, we find that the j-th part contains each level of C three times with each level of A or B, which implies that only the factor C does not enter the confounded interactions. Similarly, it will be found that the generator $(1, 1, 1)$ corresponds to the ABC interaction and corresponds to the pencil $x_1 + x_2 + x_3 = 0, 1, 2$ in $EG(3, 3)$ with the usual approach. Consider now $(3^3, 3^2)$ design, in which case we use $GF(3^2)$ to get 9 blocks. A general element of $GF(3^2)$ is $(r\theta + s)$, where $r, s = 0, 1, 2$. Let the effect vector be $(0, 1, 2)$ or $(0, 2, 1)$, as above. It will be found that factor or factors which correspond to a zero element in the generator are not confounded. Also, the generator should contain at least one element involving θ and one element out of 0, 1 or 2; otherwise, since we are working with $GF(3^2)$ but with factors at only 3 levels, the generator will not divide the treatment space into 9 equal parts. Now consider a generator of the type

$$(\zeta\theta + \lambda, \nu\theta, \mu) \tag{26}$$

This vector can be written as

$$\theta(\zeta, \nu, 0) + (\lambda, 0, \mu) \tag{27}$$

Suppose that the treatment vectors in a particular block have $(r\theta + s)$ as their product sum with (26). Then it is clear that they would give r and s respectively as products with the two component vectors of (27). If l_1, l_2, m_1, m_2 are any elements of $GF(s)$, the same block would give $\{(l_1 r + m_1 s)\theta + (l_2 r + m_2 s)\}$ as product with the generator

$$\{(l_1\zeta + m_1\lambda)\theta + l_2\zeta + m_2\lambda, l_1\nu\theta + l_2\nu, m_1\mu\theta + m_2\mu\} \tag{28}$$

and will be the block No. $[3(l_1 r + m_1 s) + (l_2 r + m_2 s) + 1]$ of the same replicate, if the generator (28) is used. The two generators (26) and

(28) are equivalent. The close connection with the usual theory is evident, the two component vectors in (27) being the two confounded pencils represented by

$$\zeta x_1 + \nu x_2 = 0, 1, 2$$
$$\lambda x_1 + \mu x_3 = 0, 1, 2.$$

It is evident that in order that no main effect is confounded, all the elements in the generator (26) should be distinct.

Consider now a p^m design in p^k blocks of p^{m-k} plots each, where p is a prime number. Here we require p^k partitions and, therefore, use $GF(p^k)$. The level vector corresponding to each factor is $(0, 1, 2, \ldots, p-1)$. We have $(p-1)$ distinct associated vectors for simple confounding. Let us use each one for one replication, getting $(p-1)$ replications in all. Now suppose we want to confound the k independent interactions represented by the k equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{21}x_2 + \ldots + a_{m_1}x_m = a_{r_1} \\ a_{12}x_1 + a_{22}x_2 + \ldots + a_{m_2}x_m = a_{r_2} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{1k}x_1 + a_{2k}x_2 + \ldots + a_{mk}x_m = a_{r_k} \end{array}\right\} \quad (29)$$

$$(r_1, r_2, \ldots, r_k = 0, 1, \ldots, p-1).$$

To the above corresponds the generator

$$\begin{array}{l} (a_{11} + a_{12}\theta + a_{13}\theta^2 + \ldots + a_{1k}\theta^{k-1}, \\ a_{21} + a_{22}\theta + a_{23}\theta^2 + \ldots + a_{2k}\theta^{k-1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1} + a_{m2}\theta + a_{m3}\theta^2 + \ldots + a_{mk}\theta^{k-1}). \end{array}$$

or

$$\left(\sum_{j=1}^{k} a_{1j}\theta^{j-1}, \sum_{j=1}^{k} a_{2j}\theta^{j-1}, \ldots, \sum_{j=1}^{k} a_{mj}\theta^{j-1}\right) \quad (30)$$

Then the generator

$$\left(\sum_{j=0}^{k-1}\sum_{i=1}^{k} \gamma_{ij}a_{1i}\theta^j, \sum_{j=0}^{k-1}\sum_{i=1}^{k} \gamma_{ij}a_{2i}\theta^j, \ldots, \sum_{j=0}^{k-1}\sum_{i=1}^{k} \gamma_{ij}a_{mi}\theta^j\right) \quad (31)$$

where γ_{ij}'s are any elements of $GF(p)$, gives the same replication (for all j) as (31) under the condition that the k vectors represented by

$$\left(\sum_{i=1}^{k} \gamma_{ij}\alpha_{1i},\ \sum_{i=1}^{k} \gamma_{ij}\alpha_{2i},\ \ldots,\ \sum_{i=1}^{k} \gamma_{ij}\alpha_{mi}\right), \tag{32}$$

j varying from 0 to $(k-1)$, are independent. This means that the vector space of rank k, with the basis given by $(\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{mj})$, j varying from 1 to k, is confounded, which corresponds to the principle of generalised interaction enunciated in this general case by Bose and Kishen.

It may be said that in a sense the generator (30) integrates the interactions confounded by the bundle of k pencils corresponding to (29) just as, for example, the moment generating function of a distribution integrates its moments.

In the case of an s^m design in s^k blocks, we can proceed in the same manner as above, remembering that the α_{ij}'s are now elements of $GF(s)$ and not of $GF(p)$.

5.2. *Kishen's Series of* $q\times 2^2$ *and* $q\times p^2$ *Designs,* q *being any Integer and* p *an Odd Prime Power*

The two series of designs given by Kishen (1958) are typical examples of non-simple confounding defined earlier. In the $q\times 2^2$ series, the 3 factors are $A(0, 1, 2, \ldots, q-1)$, $B(0, 1)$ and $C(0, 1)$. Since we want a design in $2q$ plot blocks, we use $GF(2)$. If $(1, 1, 1)$ is taken as the generator, $B(0, 1)$ and $C(0, 1)$ the associated vectors for B and C, it can be easily seen that the q replications required are obtained by taking the q associated vectors for A represented by the q unit q-vectors in $GF(2)$, namely,

$$(1, 0, 0, \ldots, 0);\quad (0, 1, 0, \ldots, 0);\quad (0, 0, 1, 0, \ldots, 0);\ \ldots$$
$$(0, 0, 0, \ldots, 0, 1) \tag{33}$$

For the $q\times p^2$ design, we use $GF(p)$, and the associated vectors of B and C are respectively

$$B(0, 1, 2, \ldots, p-1) \quad\text{and}\quad C(0, 1, 2, \ldots, p-1).$$

The set of associated vectors corresponding to A are the q-vectors in $GF(p)$ represented by

$$(\alpha_1, \alpha_0, \alpha_0, \ldots, \alpha_0);\ (\alpha_0, \alpha_1, \alpha_0, \ldots, \alpha_0);\ \ldots;\ (\alpha_0, \alpha_0, \ldots, \alpha_0, \alpha_1)$$
$$(\alpha_2, \alpha_0, \alpha_0, \ldots, \alpha_0);\ (\alpha_0, \alpha_2, \alpha_0, \ldots, \alpha_0);\ \ldots;\ (\alpha_0, \alpha_0, \ldots, \alpha_0, \alpha_2)$$

....

....

....

$$(\alpha_{(p-1)/2}, \alpha_0, \alpha_0, \ldots, \alpha_0);\ (\alpha_0, \alpha_{(p-1)/2}, \alpha_0, \ldots, \alpha_0);\ \ldots;\ (\alpha_0, \alpha_0, \ldots, \alpha_0, \alpha_{(p-1)/2}). \tag{34}$$

It can be seen that if the vectors in (33) and each of the $(p-1)/2$ sets of vectors in (34) are arranged in the form of a square, as under,

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \qquad (35)$$

the element 1 of $GF(2)$ falls once in each row and once in each column. Comparing it with the latin square

$$\begin{bmatrix} A_1 & A_2 & A_3 & \ldots & A_q \\ A_q & A_1 & A_2 & \ldots & A_{q-1} \\ A_{q-1} & A_q & A_1 & \ldots & A_{q-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_2 & A_3 & A_4 & \ldots & A_1 \end{bmatrix} \qquad (36)$$

we find that here the letter A has been replaced by 1 and the others by 0.

This makes possible a generalization of this approach for construction of $q \times 2^2$ and $q \times p^2$ designs with less loss of information on BC. For example, if $A_1, A_2, \ldots, A_t$ in the second square are replaced by 1 and $A_{t+1}, A_{t+2}, \ldots, A_q$ by 0, and the q rows of the resulting square are taken as associated vectors of A, we shall get a $q \times 2^2$ design in $2q$ plot blocks, the loss of information on BC being $(q-2t)^2/q^2$. A similar approach with the $q \times p^2$ series can be made and can be utilized to construct the $5 \times 3 \times 3$ design in 15 plot blocks.

5.3. *Certain Factorial Designs Using b.i.b. Property*

Let s be a prime power and a any integer. Let there be two factors A and B, each at $a \times s$ levels. We can divide the total number of a^2s^2 treatment combinations into $a \times s$ blocks of $a \times s$ plots each in such a way that no main effect is confounded. Suppose these $a \times s$ blocks confounding $(as-1)$ *d.f.* belonging to interaction AB are $X_1, X_2, \ldots, X_{as}$. Now, suppose, a balanced incomplete block design (b.i.b.d.) exists with $v = as$, and block size $k < v$. Then immediately we get a confounded factorial design $as \times as$ in blocks of size kas, partially confounding $(as - 1)$ *d.f.* of interaction AB. Preferably k should be small. In particular, we can always have $k = 2$, the b.i.b.d. in this case being $v = as$, $b = as\ (as - 1)/2$, $k = 2$, $r = (as - 1)$, $\lambda = 1$, giving $as \times as$ design in $2\,as$ plot blocks.

Alternatively, the b.i.b.d., $v = as$, $b = s\,(as - 1)$, $k = a$, $r = (as - 1)$, $\lambda = a - 1$, can also be considered if one exists.

When $a = 2$ and $s = 3$, we get the b.i.b.d. $v = 6$, $b = 15$, $k = 2$, $r = 5$ and $\lambda = 1$, which gives the 6×6 design in 12 plot blocks shown below.

TABLE IV

6×6 *Balanced design in* 12-*plot blocks*

Replication ..		I			II			III			IV			V	
Block No. ..	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	X_1	X_3	X_5	X_1	X_2	X_4	X_1	X_2	X_3	X_1	X_2	X_3	X_1	X_2	X_4
	X_2	X_4	X_6	X_3	X_5	X_6	X_4	X_6	X_5	X_5	X_4	X_6	X_6	X_3	X_5

Here the treatment combinations included in the set X_i are those which satisfy the equation

$$x_1 + x_2 = i \bmod 6 \qquad (37)$$

$$(i = 0,\ 1,\ 2,\ 3,\ 4,\ 5).$$

It will be seen that this method gives designs in which the total loss of information due to confounding is less than in the designs obtained by using the $a \times s$ sets $X_1, X_2, \ldots, X_{as}$ as blocks.

The method is particularly useful in symmetrical or asymmetrical designs in which the number of levels of each factor is not large, and the block size can be increased without appreciable increase in error. For example, we can construct a 5×5 factorial design in 10 plot blocks in 4 replications by taking a b.i.b.d. with $v = 5$, $b = 10$, $k = 2$, $r = 4$ and $\lambda = 1$. Similarly, we can construct a 7×4 factorial design in 12 plot blocks in 3 replications by considering the b.i.b.d., $v = 7$, $b = 7$, $k = 3$, $r = 3$ and $\lambda = 1$, on the 7 sets obtained by using $GF(7)$ along with the associated vectors $A\,(0, 1, 2, \ldots, 6)$, $B\,(0, 1, 2, 3)$ and the generator say (p, q) where p, q are non-zero elements. It is noticeable that by this procedure the number of replications required for balancing is only 3 as against 6 replications in the design derivable directly from the Galois field with block of 4 plots.

5.4. $(p_1, p_2)^m$ *Designs in Blocks of* $p_1^{m-r_1} p_2^{m-r_2}$ *Plots,* p_1, p_2 *being any Prime Powers and* $r_1, r_2 \leqslant m$.

The procedure is simple. First, we form $p_1^{r_1}$ blocks by considering a particular generator the elements of which belong to $GF(p_1^{r_1})$. The associated vector for each factor may be such that it is divisible into p_2 sets of elements, each set containing p_1 distinct elements belonging to $GF(p_1)$. At the next stage, we similarly consider $GF(p_2^{r_2})$ for further dividing each block, for each of which we take the same generator. The associated vector for each factor in this case consists of p_2 distinct elements belonging to $GF(p_2)$, one element corresponding to all elements in one of the sets out of the p_2 sets defined above for the earlier associated vector. The procedure will be illustrated by deriving a 6×6 design in 6 plot blocks.

The associated vectors for A and B at the first stage may be taken as (0, 1, 2, 0, 1, 2), and the generator as (1, 1). We use $GF(3)$ since $6 = 3 \times 2$. At the second stage, we use $GF(2)$ with the generator (1, 1) and associated vector for both A and B as (0, 0, 0, 1, 1, 1). This gives a set of 6 blocks for the first replication. To this we may add another replication obtained by taking (1, 2) as the generator at the first stage. The two replications together provide a balanced design, which is given in Table V.

TABLE V

6×6 Design in 6-plot blocks

Replication ..	I						II					
Blocks ..	X_0	X_1	X_2	X_3	X_4	X_5	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5
	00	01	02	03	04	05	00	01	02	03	04	05
	12	10	11	15	13	14	11	12	10	14	15	13
	21	22	20	24	25	23	22	20	21	25	23	24
	33	34	35	30	31	32	33	34	35	30	31	32
	45	43	44	42	40	41	44	45	43	41	42	40
	54	55	53	51	52	50	55	53	54	52	50	51

In this design the single degree of freedom for AB corresponding to the contrast $(a_5 + a_4 + a_3 - a_2 - a_1 - a_0)(b_5 + b_4 + b_3 - b_2 - b_1 - b_0)$ is totally confounded in both the replications. Further, 8 more degrees of freedom belonging to AB are partially confounded, on which the loss of information is $\frac{1}{2}$. The total loss of information is $1 + 8 \times \frac{1}{2} = 5$, which is equal to the number of degrees of freedom confounded in each replicate so that the design is a balanced arrangement.

Balancing for the case $m = 2$ would be achieved in $(p_1 - 1)(p_2 - 1)$ replications, which would be obtained by varying the second element of the first stage generator over $(p_1 - 1)$ non-null elements of $GF(p_1)$ and further for each of these cases by varying the second element of the second stage generator over the $(p_2 - 1)$ non-null elements of $GF(p_2)$. Balanced designs of the type $(p_1, p_2, \ldots, p_k)^m$ in blocks of $p_1^{m-r_1}$, $p_2^{m-r_2}$ $\ldots, p_k^{m-r_k}$ plots, where $r_1, r_2, \ldots, r_k \leqslant m$, can be constructed in a similar manner in $(p_1 - 1)(p_2 - 1) \ldots (p_k - 1)$ replications (provided $r_j \leqslant p_j - 1$; $j = 1, 2, \ldots, k$).

5.5. *Balanced Asymmetrical Designs with Reduced Number of Replications*

Firstly, let us consider three-factor designs of the type $s_1 \times s_2 \times s_3$ where both s_1 and s_2 are prime powers. Let $s_1 \geqslant s_2 \geqslant s_3$. From Section 4, we know that if $s_1 = s_2$, we can construct a design in blocks of $s_2 s_3$ plots balanced in $(s_1 - 1)$ replications. In case $s_1 \neq s_2$, the method given there provides a balanced design in $(s_1 - 1)^2$ replications. We may, therefore, use a modified method in such a case.

Consider, first, $GF(s_2)$. Let the associated vectors corresponding to A_2 and A_3 be $(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{s_3-1})$ and $(\alpha_0, \alpha_1, \ldots, \alpha_{s_3-1})$ respectively. Taking a generator, say, (α_1, α_1), in $GF(s_2)$, we can form one replication of s_2 sets of treatment combinations of the factors A_2 and A_3, each set containing s_3 treatment combinations. The j-th set will obviously contain those combinations of levels of A_2 and A_3, the elements of the associated vectors corresponding to which give a sum product α_{j-1} when multiplied by the generator. Let us denote these sets by $X_0, X_1, X_2, \ldots, X_{s_2-1}$ respectively.

Proceeding, as above, we can similarly construct a design in s_2 plot blocks in $(s_1 - 1)$ replications by considering the two factors A_1 and A_2 only with associated vectors $(\alpha_0, \alpha_1, \ldots, \alpha_{s_1-1})$ and $(\alpha_0, \alpha_1, \ldots, \alpha_{s_2-1})$ in $GF(s_1)$ and taking the $(s_1 - 1)$ generators represented respectively by (α_1, α_r), where r varies from 1 to $s_1 - 1$. This is simply an $s_1 \times s_2$ design; and to extend it to the $s_1 \times s_2 \times s_3$ design, we may now replace the j-th level of the factor A_2 in this design by the set X_j defined above containing s_3 combinations of levels of A_2 and A_3.

As an illustration of this procedure, consider the $5 \times 3 \times 2$ design. The sets X_j obtained would be

X_0 : b_0c_0, b_2c_1

X_1 : b_1c_0, b_0c_1

X_2 : b_2c_0, b_1c_1

These sets, when combined with the 5 levels of A_1 with the help of $GF(5)$, will give the following $5 \times 3 \times 2$ design:

TABLE VI

$5 \times 3 \times 2$ *Design in 6-plot blocks*

	Replication I					Replication II				
Block No. ..	1	2	3	4	5	6	7	8	9	10
	000	100	200	300	400	000	300	100	400	200
	021	121	221	321	421	021	321	121	421	221
	311	411	011	111	211	411	211	011	311	111
	320	420	020	120	220	420	220	020	320	120
	410	010	110	210	310	210	010	310	110	410
	401	001	101	201	301	201	001	301	101	401

	Replication III					Replication IV				
Block No. ..	11	12	13	14	15	16	17	18	19	20
	000	200	400	100	300	000	400	300	200	100
	021	221	421	121	321	021	421	321	221	121
	111	311	011	211	411	211	111	011	411	311
	120	320	020	220	420	220	120	020	420	320
	310	010	210	410	110	110	010	410	310	210
	301	001	201	401	101	101	001	401	301	201

In the above design, the main effects and interactions confounded are A, AB (4 *d.f.*) and ABC. It is noticeable that AC is not confounded although the number of combination of levels of A and C is 10 and the block size is 6. The loss of information on each of 4 *d.f.* of A is 1/6; that on 4 *d.f.* of AB is 0 and on each of the remaining 4 *d.f.* of AB is 5/24; and, finally, on each of 4 *d.f.* of ABC, the loss is 5/24 and on each of the remaining 4 *d.f.* of ABC, it is 10/24. The total loss of information is, therefore, 4, so that the design is balanced.

Designs of the type $7\times3\times2$ (in 6-plot blocks involving 6 replications), $8\times3\times2$ (in 6-plot blocks involving 7 replications), $5\times4\times3$ (in 12-plot blocks involving 4 replications), $7\times4\times3$ (in 12-plot blocks involving 6 replications), $7\times5\times3$ (in 15-plot blocks involving 6 replications), etc., can be easily constructed by the above method.

For the construction of a four-factor design, say, $5\times3\times2\times2$ in 12-plot blocks, we can proceed exactly as in the above manner, combining first A_4 and A_3 and making two sets with two treatment combinations in each set; combining these two sets with A_2 in $GF(3)$, making 3 new sets of 4 treatment combinations each; and finally making five sets of 12 treatment combinations each by combining the 3 sets formed in the last case with the factor A_1, using $GF(5)$. The design so obtained will be balanced in 4 replications. Obviously, the design can be represented by the plan given above for the $5\times3\times2$ design with the modification that in place of the two levels of C, namely, 0 and 1, we have now to put respectively two sets of levels of C and D, namely, c_0d_0, c_1d_1 and c_0d_1, c_1d_0. It will be found that in the design so generated, the main effect A will be confounded and also the interactions AB and $ABCD$.

The above procedure can be easily and usefully generalized for the construction of the general asymmetrical factorial design $s_1\times s_2\times\ldots\times s_m$ (where $s_1 \geqslant s_2 \geqslant s_3 \geqslant \ldots \geqslant s_m$, and $s_1, s_2, \ldots, s_{m-1}$ are all prime powers) in blocks of $s_2\times\ldots\times s_m$ plots. Further, having obtained such a design, we can reduce the block size one step further to $s_3\times\ldots\times s_m$ by splitting all the $s_2\times\ldots\times s_m$ treatment combinations in a block into s_2 sets of $s_3\times s_4\times\ldots\times s_m$ treatment combinations each by the use of $GF(s_2)$ over the factors $A_2A_3\ldots A_m$. It has to be remembered that for doing this the same generator is to be used for all the blocks.

As an illustration, the $5\times3\times3\times2$ design in blocks of 6 plots will be presented.

In order to avoid complete confounding over AB, we first combine B and C, and make the one plot sets Z_{ij} given by

$$Z_{ij} = b_r c_t$$

where

$$r + t = i \bmod 3$$
$$r + 2t = j \bmod 3$$

Next, we combine Z_{ij} with D, and get the sets X_i

$$X_0 : Z_0 d_0, Z_2 d_1$$
$$X_1 : Z_1 d_0, Z_0 d_1$$
$$X_2 : Z_2 d_0, Z_1 d_1$$

where Z_i consist of Z_{ij} for all j. We then get a $5 \times 3 \times 3 \times 2$ design in $3 \times 3 \times 2$ plot blocks by combining the X's with the factor A. To get a design in blocks of 6 plots, we simply put the treatments with the same j in Z_j in the same block, and those with separate j's in separate blocks. This will give a design in 4 replications. However, in order to have a balanced design, we shall have to use four more replications obtained by separating Z_i's with respect to i in the same way as was done with j. The total number of replications required for balancing in this case is $(5-1)(3-1) = 8$, and is, in general, $(s_1 - 1)(s_2 - 1)$ since we use two Galois fields, each once, and make $s_1 s_2$ blocks per replication. The $5 \times 3 \times 3 \times 2$ design is given in Table VII, where the loss of information is also shown. The total loss of information is seen to be 14, or one less than the number of blocks per replication, so that the design is balanced. This design can be easily generalized to $s \times 3 \times 3 \times 2$ where s is a prime power.

TABLE VII

$5 \times 3 \times 3 \times 2$ Design in 6-plot blocks

Combinations of B, C, D	Level of A																				Combinations of B, C, D for four exactly similar replications
	Replication I					Replication II					Replication III					Replication IV					
Block No. ..	1	4	7	10	13	16	19	22	25	28	31	34	37	40	43	46	49	52	55	58	
000, 111 ..	0	1	2	3	4	0	3	1	4	2	0	2	4	1	3	0	4	3	2	1	000, 221
220, 001 ..	4	0	1	2	3	2	0	3	1	4	3	0	2	4	1	1	0	4	3	2	110, 001
110, 221 ..	3	4	0	1	2	4	2	0	3	1	1	3	0	2	4	2	1	0	4	3	220, 111

TABLE VII (*Contd.*)

Combinations of B, C, D	Level of A																				Combinations B, C, D of for four exactly similar replications
	Replication I					Replication II					Replication III					Replication IV					
Block No. ..	2	5	8	11	14	17	20	23	26	29	32	35	38	41	44	47	50	53	56	59	
210, 021 ..	0	1	2	3	4	0	3	1	4	2	0	2	4	1	3	0	4	3	2	1	210, 101
100, 211 ..	4	0	1	2	3	2	0	3	1	4	3	0	2	4	1	1	0	4	3	2	020, 211
020, 101 ..	3	4	0	1	2	4	2	0	3	1	1	3	0	2	4	2	1	0	4	3	100, 021
Block No. ..	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	
120, 201 ..	0	1	2	3	4	0	3	1	4	2	0	2	4	1	3	0	4	3	2	1	120, 011
010, 121 ..	4	0	1	2	3	2	0	3	1	4	3	0	2	4	1	1	0	4	3	2	200, 121
200, 011 ..	3	4	0	1	2	4	2	0	3	1	1	3	0	2	4	2	1	0	4	3	010, 201

Total number of replications = 8

Loss of information

		Relative loss per *d.f.*	Total loss
A	..	1/6	4/6
AB	..	5/48	5/6
AC	..	5/48	5/6
$BC\,(I)$	..	1	12/6 (Totally confounded)
$BC\,(J)$	..	0	0
$ABC\,(I)$	..	8/48	8/6
$ABC\,(J)$	..	5/48	5/6
ABD	..	15/48	15/6
ACD	..	15/48	15/6
$ABC\,(I)\,D$	..	15/48	15/6
$ABC\,(J)\,D$	..	0	0
BCD, CD, AD, BD, B, C, D	..	0	0

Total Loss .. 14 = No. of blocks per replication − 1.

6. Some Further Balanced Designs

6.1. *Derivation of One Balanced Design from Another*

Suppose there already exists a balanced design $s_1 \times s_2 \times \ldots \times s_m$ in blocks of k plots (k may be $s_2 \times s_3 \times \ldots \times s_m$), and we wish to derive the design for the $a_1 s_1 \times a_2 s_2 \times \ldots \times a_m s_m$ factorial experiment from it, where $a_1, a_2, \ldots, a_m$ are any positive integers. Also, suppose that in the construction of the given design $s_1 \times s_2 \times \ldots \times s_m$ we had used for the factor A_i, an associated vector

$$(Z_1, Z_2, \ldots, Z_{si})$$

where the Z_j's are any elements, not necessarily distinct, of the Galois field used for the purpose of generating the blocks. Then, for the construction of the $a_1 s_1 \times \ldots \times a_m s_m$ design, we may simply take for the A_i an associated vector of the form

$$(Z_1, Z_2, \ldots, Z_{si};\; Z_1, Z_2, \ldots, Z_{si};\; \ldots;\; Z_1, Z_2, \ldots, Z_{si})$$

each Z_j being repeated a_i times in this vector. Such associated vectors for A_i should be used in the new design corresponding to all associated vectors which were used for the factor A_i when the given design $s_1 \times s_2 \times \ldots \times s_m$ was constructed. The block size in the derived design will be $a_1 \times a_2 \times \ldots \times a_m \times k$. The block size can be further reduced to any extent by repeated use of suitable Galois fields. For this purpose factorisation of a's into prime powers may also be done. This procedure of derivation of designs with non-prime levels gives a number of useful designs.

As an illustration, we derive the $6 \times 2 \times 2$ design from the $3 \times 2 \times 2$ design. The associated vectors that we use for the $3 \times 2 \times 2$ design are (0, 1) for B and C and (0, 0, 1); (0, 1, 0); (1, 0, 0) for A to be used respectively for the three replications in which balancing is achieved. All these vectors are in $GF(2)$. For the $6 \times 2 \times 2$ designs, we use the same vectors for B and C and for A we use (0, 0, 1; 0, 0, 1); (0, 1, 0; 0, 1, 0) and (1, 0, 0; 1, 0, 0) respectively for the three replications. This gives the $6 \times 2 \times 2$ design shown in Table VIII, in which X_0 and X_1 denote respectively the sets $(b_0c_0,\ b_1c_1)$ and $(b_1c_0,\ b_0c_1)$. In this design, the total loss of information on BC is 1/9 and that on the two confounded *d.f.* of ABC is 8/9, so that the total loss is unity and the design is balanced.

As already mentioned, the above procedure can be used for construction of all designs irrespective of the number and type of Galois

TABLE VIII

$6 \times 2 \times 2$ *Design in* 12-*plot blocks*

		Replication I		Replication II		Replication III	
Block No.	..	1	2	3	4	5	6
Level of A		Levels of B and C					
a_0	..	X_1	X_0	X_0	X_1	X_0	X_1
a_1	..	X_1	X_0	X_0	X_1	X_0	X_1
a_2	..	X_0	X_1	X_1	X_0	X_0	X_1
a_3	..	X_0	X_1	X_1	X_0	X_0	X_1
a_4	..	X_0	X_1	X_0	X_1	X_1	X_0
a_5	..	X_0	X_1	X_0	X_1	X_1	X_0

fields utilized for getting the block size k. Thus, for a 3×2^3 design in 6-plot blocks, we use $GF(2)$ twice with the associated vectors of A as given above for the 3×2^2 design. From this design we can, therefore, immediately derive the 6×2^3 design simply by using the associated vector for A as given above for the 6×2^2 design.

A similar procedure is adopted in those cases where two different Galois fields are to be used. For example, consider the $6 \times 6 \times 2$ design in 12-plot blocks. First, we divide the 72 treatment combinations into two sets of 36 each by using $GF(2)$ together with (i) a generator of the form (1, 1, 1), (ii) the associated vector (0, 0, 0, 1, 1, 1) for both A and B and (0, 1) for C. At the second stage, we use $GF(3)$ and take (0, 1, 2, 0, 1, 2) as the associated vectors of A and B and (0, 1) as the associated vector of C along with two generators (1, 1, 1) and (1, 2, 1) for getting two different replications, which will provide the balanced design given in Table IX. In this Table, c_0 and c_1 denote the two levels of C and X_i and Y_i ($i = 0, 1, \ldots 5$) the sets of combinations of levels of A and B as given in the plan for the 6×6 design. It can be easily seen that in this design, the interactions AB and ABC are confounded, the total loss of information being respectively 2 and 3.

TABLE IX

$6 \times 6 \times 2$ *Design in* 12-*plot blocks*

Block No.	..	1	2	3	4	5	6
Replication I	..	X_0c_0 X_2c_1	X_1c_0 X_0c_1	X_2c_0 X_1c_1	X_3c_0 X_5c_1	X_4c_0 X_3c_1	X_5c_0 X_4c_1
Block No.	..	7	8	9	10	11	12
Replication II	..	Y_0c_0 Y_1c_1	Y_2c_0 Y_0c_1	Y_1c_0 Y_2c_1	Y_3c_0 Y_4c_1	Y_4c_0 Y_5c_1	Y_5c_0 Y_3c_1

The $s^2 \times t$ design in $s \times t$ plot blocks, where s is non-prime, can be constructed by the methods of this Section as a particular case. Some of the other useful asymmetrical designs which can be constructed in an optimum manner by use of the above methods are 6×4, $6 \times 4 \times 3$, $6 \times 4 \times 2$, $6 \times 4 \times 4$, $4 \times 3 \times 2$, $6 \times 3 \times 2$, $3 \times 3 \times 3 \times 4$, $6 \times 3 \times 3 \times 2$, etc.

6.2. *Further Use of b.i.b.d. Property in Obtaining Confounded Designs*

Almost all types of factorial designs arising in practice can be constructed in an optimum manner by appropriately using the methods discussed so far. Still another method, which may be of use in certain situations and which serves to indicate the connection between balancing in asymmetrical factorial experiments and balanced incomplete block designs, will now be discussed.

Let us consider the construction of a $7 \times 2 \times 2$ design. By the use of hypersurfaces or associated vectors, we can construct a design in 4-plot blocks with 6 replications. In this design, the main effect A is confounded, which is not desirable. The design belonging to Kishen's series $q \times 2^2$ discussed in Section (5.2), taking $q = 7$, is to be preferred in this case, as only the interactions BC and ABC are partially confounded in this design. However, the loss of information on BC in this case is 25/49. An alternative approach for obtaining an optimum design in this case is by use of the b.i.b.d. property and will now be discussed.

Each replication will be divided into two blocks of $2q$ plots each. Let X_0 and X_1 denote the sets (b_0c_0, b_1c_1) and (b_1c_0, b_0c_1) of treatment combinations respectively. Consider a single replication. Block No. 1 of this replication will contain the treatment combinations $[a'] X_0$ and $[a''] X_1$, where a' and a'' represent two exhaustive groups for the levels of A and Block No. 2 will have the complement of this. If a_1' and a_2' are any two levels in a', and a_1'', a_2'' belong to a'', then obviously $(a_1' - a_2')(X_1 - X_0)$, $(a_1'' - a_2'')(X_1 - X_0)$ and $[(a_1' - a_2') \pm (a_1'' - a_2'')]$ $(X_1 - X_0)$, all of which belong to interaction ABC, will not be confounded. However, contrasts like $(a_1 - a_1'')(X_1 - X_0)$ will be confounded. The question is how to determine a' and a'' so that a balanced design is obtained.

A solution to this problem may be found by considering the possibility of selecting the set a' in the different replications in such a manner that the levels of A included in a' form a balanced incomplete block design. Since, in that case, every pair of levels of A will occur on equal number of times with X_0 or X_1, every contrast of the type $(a_i \pm a_j)$ $(X_1 - X_0)$, $(i \neq j = 1, 2, \ldots, q)$ will be partially confounded to the same extent. Thus, both BC and ABC will be estimable, and the design will be a balanced arrangement.

The loss of information on BC depends on the number of levels in the sets a' and a''. For $7 \times 2 \times 2$ design, we can take 3 levels for a' and 4 for a'', and obtain the design given in Table X.

TABLE X

$7 \times 2 \times 2$ *Design in* 14-*plot blocks*

A	I	II	III	IV	V	VI	VII
a_0 ..	X_0X_1	X_0X_1	X_0X_1	X_1X_0	X_1X_0	X_1X_0	X_1X_0
a_1 ..	X_0X_1	X_1X_0	X_1X_0	X_0X_1	X_0X_1	X_1X_0	X_1X_0
a_2 ..	X_0X_1	X_1X_0	X_1X_0	X_1X_0	X_1X_0	X_0X_1	X_0X_1
a_3 ..	X_1X_0	X_0X_1	X_1X_0	X_0X_1	X_1X_0	X_0X_1	X_1X_0
a_4 ..	X_1X_0	X_0X_1	X_1X_0	X_1X_0	X_0X_1	X_1X_0	X_0X_1
a_5 ..	X_1X_0	X_1X_0	X_0X_1	X_0X_1	X_1X_0	X_1X_0	X_0X_1
a_6 ..	X_1X_0	X_1X_0	X_0X_1	X_1X_0	X_0X_1	X_0X_1	X_1X_0

In the above design, loss of information on BC is 1/49 and on each of the six *d.f.* of ABC is 8/49, so that the total loss of information is 1. The design is, therefore, balanced.

We shall now proceed to the general case and consider a $q \times p^2$ design, where p is an odd prime power and q any integer. (It can be easily seen that the method holds when p is of the form 2^n). Consider a generator of the type, say, $(\alpha_1, \alpha_1, \alpha_1)$, where α_1 is an element of $GF(p)$. Let k_i $(i = 1, 2, \ldots, p)$ be any integers such that $0 \leqslant k_i \leqslant q-1$, and $\sum_{i=1}^{p} k_i = q$, so that these divide the total number of levels of A in p groups, of which one or more groups may have no elements in them. Also, suppose that for $i = 1, 2, \ldots, p$, balanced incomplete block designs exist with parameters $v = q$, $k = k_i$ and suitable values of b, r_i and λ_i and can be superimposed on one another so as to give b blocks of q plots each. Then we take an associated vector of the form $(\alpha_0, \alpha_1, \ldots, \alpha_{p-1})$ for B and C, and b different associated vectors for A, one vector corresponding to each replication, such that each of the elements in the j-th group (containing k_j elements), is α_{ij}, an element of $GF(p)$. A balanced design in b replicates will then be obtained by using the generator with each of the b associated vectors for A and the common associated vector for B and C.

As an illustration, consider the $7 \times 3 \times 3$ design in 21-plot blocks. We shall consider the three sets X_0, X_1, X_2 of treatment combinations for the interaction BC defined below:

X_0 : b_0c_0, b_2c_1, b_1c_2
X_1 : b_1c_0, b_0c_1, b_2c_2
X_2 : b_2c_0, b_1c_1, b_0c_2

Here $q = 7$. Also, if we take $k_1 = 3$, $k_2 = 4$ and $k_3 = 0$, we shall find that two b.i.b. designs with $v = 7$ exist, which are superimposable, as shown in Table XI.

TABLE XI

Two b.i.b. designs with $v = 7$

B.I.B.D. No.	..	1			2			
Block No.								
1	..	1	2	3	7	6	5	4
2	..	1	4	5	7	3	2	6
3	..	1	6	7	5	4	2	3
4	..	2	4	6	5	3	1	7
5	..	2	5	7	6	4	3	1
6	..	3	4	7	6	2	1	5
7	..	3	5	6	7	4	1	2

We can, therefore, form the 7 different associated vectors for A corresponding to the 7 replications in the design. Thus, for the 5th replication, the associated vector for A will be $(\alpha_{i_2}, \alpha_{i_1}, \alpha_{i_2}, \alpha_{i_2}, \alpha_{i_1}, \alpha_{i_2}, \alpha_{i_1})$ when the α_{ij} are elements of $GF(3)$. Here $\alpha_{i_1} = 0$ and $\alpha_{i_2} = 1$. By using the generator (1, 1, 1), we shall then obtain the design given in Table XII.

TABLE XII

7×3×3 Design in 21-plot blocks

a_0	..	$X_0X_1X_2$	$X_0X_1X_2$	$X_0X_1X_2$	$X_2X_0X_1$	$X_2X_0X_1$	$X_2X_0X_1$	$X_2X_0X_1$
a_1	..	$X_0X_1X_2$	$X_2X_0X_1$	$X_2X_0X_1$	$X_0X_1X_2$	$X_0X_1X_2$	$X_2X_0X_1$	$X_2X_0X_1$
a_2	..	$X_0X_1X_2$	$X_2X_0X_1$	$X_2X_0X_1$	$X_2X_0X_1$	$X_2X_0X_1$	$X_0X_1X_2$	$X_0X_1X_2$
a_3	..	$X_2X_0X_1$	$X_0X_1X_2$	$X_2X_0X_1$	$X_0X_1X_2$	$X_2X_0X_1$	$X_0X_1X_2$	$X_2X_0X_1$
a_4	..	$X_2X_0X_1$	$X_0X_1X_2$	$X_2X_0X_1$	$X_2X_0X_1$	$X_0X_1X_2$	$X_2X_0X_1$	$X_0X_1X_2$
a_5	..	$X_2X_0X_1$	$X_2X_0X_1$	$X_0X_1X_2$	$X_0X_1X_2$	$X_2X_0X_1$	$X_2X_0X_1$	$X_0X_1X_2$
a_6	..	$X_2X_0X_1$	$X_2X_0X_1$	$X_0X_1X_2$	$X_2X_0X_1$	$X_0X_1X_2$	$X_0X_1X_2$	$X_2X_0X_1$

It will be seen that in the above design, the interactions BC and ABC are confounded.

The above procedure can be used to obtain other asymmetrical designs also, for example, the 7×3×2 design in 14-plot blocks.

7. Analysis of Balanced Designs

Two general methods of analysing partially confounded designs, which are balanced, will now be briefly discussed to enable the interested reader to work out formulæ for analysis in the case of any specific design.

7.1. *The* Q_j *Method*

In any general design where

$N_{i.}$ = number of plots in the ith block ($i = 1, \ldots, b$),

$N_{.j}$ = number of replications of the j-th treatment ($j = 1, \ldots, t$),

n_{ij} = number of times the j-th treatment occurs in the ith block, and where t_j denotes the effect of the j-th treatment, it is known (Kempthorne, 1952) that the normal equations for estimating the t_j can be written in the form

$$\left(N_{.j} - \sum_i \frac{n_{ij}^2}{N_{i.}}\right) t_j - \sum_{k \neq j} \left(\sum_i \frac{n_{ij} n_{ik}}{N_{i.}}\right) t_k$$
$$= Q_j,\ j = 1, 2, \ldots, t \tag{38}$$

where

$$Q_j = Y_{.j} - \sum \frac{n_{ij}}{N_{i.}} Y_{i.} \tag{39}$$

Since the block size and number of replicates of a treatment for all the designs discussed in this paper are constant, we shall take $N_{i.} = k$ and $N_{.j} = r$. Thus, the normal equations in (38) reduce to

$$r\left(1 - \frac{1}{k}\right) t_j - \frac{1}{k}(\text{sum of } t\text{'s that occur in a block with } t_j) = Q_j \tag{40}$$

The quantities Q_j are the well-known adjusted yields for a treatment, and may be calculated in an easy and straightforward way for all designs, as indicated in the Table below:

Treatment No.	Total yield from all replications	Blocks in which the j-th treatment occurs	Total yield of all blocks in which j-th treatment occurs	S/k	$Q_j = Y_j - (1/k) S_j$
1	Y_1	$B_{11}, B_{12}, \ldots, B_{1r}$	S_1	S_1/k	Q_1
2	Y_2	$B_{21}, B_{22}, \ldots, B_{2r}$	S_2	S_2/k	Q_2
.	.	. . .	.	.	.
.	.	. . .	.	.	.
.	.	. . .	.	.	.
j	Y_j	$B_{j1}, B_{j2}, \ldots, B_{jr}$	S_j	S_j/k	Q_i
t	Y_t	$B_{t1}, B_{t2}, \ldots, B_{tr}$	S_t	S_t/k	Q_t

The blocks B_{ij} are not all distinct for different i and j.

Now if $\Sigma \lambda_i t_i$ is a treatment contrast which we wish to estimate, it can be estimated by $1/Ir \, \Sigma \lambda_i Q_i$ where I is the relative information on the contrast. For obtaining the relative information, we first calculate the expectation of $\Sigma \lambda_j Q_j$, where the expected value of Q_j is as shown on the left-hand side of equation (40). Then, if C_j is the coefficient of t_j in this expected value, we shall find that if $\Sigma \lambda_i t_i$ is estimable from the full design, the value of C_j/λ_j will be a constant C for all j. Then, the relative information required is given by

$$I = \frac{C}{r} \tag{41}$$

where, as above, r is the total number of replications. Having calculated I, the sum of squares corresponding to this contrast will be given by

$$\frac{1}{Ir} \times \frac{1}{\Sigma \lambda_i^2} (\Sigma \lambda_i Q_i)^2 \tag{42}$$

where, for this purpose, λ_i's should be $+1$, -1 or zero.

As an illustration, consider the analysis of the 4×3^2 design, the plan of which is given in Table XIII.

TABLE XIII

Plan of 4×3^2 Design in 12-plot blocks

	Level of B and C					
Block No. ..	1	2	3	4	5	6
Level of A						
a_0 ..	I_0	I_1	I_2	I_0	I_1	I_2
a_1 ..	I_0	I_1	I_2	I_0	I_1	I_2
a_2 ..	I_1	I_2	I_0	I_2	I_0	I_1
a_3 ..	I_1	I_2	I_0	I_2	I_0	I_1

The I's denote as usual the well-known I sets, corresponding to the interaction $BC(I)$. It will be found that the relative loss of information on the interaction $BC(I)$ is $\frac{1}{4}$ and that on $(a_3 + a_2 - a_1 - a_0)\, BC(I)$ components of ABC interaction is $\frac{3}{4}$. Two independent comparisons belonging to $BC(I)$ are $L_1 = (a_3 + a_2 + a_1 + a_0)(I_2 - I_0)$ and $L_2 =$

$(a_3 + a_2 + a_1 + a_0)(I_2 - 2I_1 + I_0)$. The estimates of L_1 and L_2 will be found to be $\hat{L}_1 = 4/3\,(Q_{I_2} - Q_{I_0})$, $\hat{L}_2 = 4/3\,(Q_{I_2} - 2Q_{I_1} + Q_{I_0})$ where Q_{I_0} is the sum of the Q's for all the nine treatment combinations contained in the set $a_i I_0$ $(i = 0, 1, 2, 3;\ I_0 = b_0c_0, b_2c_1, b_1c_2)$, with similar definitions for QI_1 and QI_2. Similarly, if $L_3 = (a_3 + a_2 - a_1 - a_0)(I_2 - I_0)$ and $L_4 = (a_3 + a_2 - a_1 - a_0)(I_2 - 2I_1 + I_0)$, we shall have $\hat{L}_3 = 4\,(Qa_3I_2 + Qa_2I_2 - Qa_1I_0 - Qa_0I_0 - Qa_3I_0 - Qa_2I_0 + Qa_1I_0 + Qa_0I_0)$, with a similar expression for $\hat{L}_4$. The sum of squares for $BC\,(I)$ will be

$$\frac{1}{9}(Q_{I_2} - Q_{I_0})^2 + \frac{1}{27}(Q_{I_2} - 2Q_{I_1} + Q_{I_0})^2.$$

7.2. *Yates's Method*

This method, which has been suggested by Yates (1937), has been used by him and Li (1944) for the analysis of balanced asymmetrical factorial designs. The method can be utilized for the analysis of the balanced designs discussed in this paper. However, it is particularly appropriate for the analysis of Kishen's series of designs presented in Section (5.2). For illustration, we shall now give briefly the method of analysis of the $q \times 2^2$ design in blocks of $2q$ plots, the plan for which is shown in Table XIV.

TABLE XIV

Plan of $q \times 2^2$ Design

		B_{11}	B_{12}	B_{21}	B_{22}	B_{31}	B_{32}	.	.	B_{q1}	B_{q2}
Level of A											
a_0	..	X_0	X_1	X_1	X_0	X_1	X_0	.	.	X_1	X_0
a_1	..	X_1	X_0	X_0	X_1	X_1	X_0	.	.	X_1	X_0
a_2	..	X_1	X_0	X_1	X_0	X_0	X_1	.	.	X_1	X_0
.		.	.	.	.	.	.	.	.	.	.
.		.	.	.	.	.	.	.	.	.	.
.		.	.	.	.	.	.	.	.	.	.
a_{q-1}	..	X_1	X_0	X_1	X_0	X_1	X_0	.	.	X_0	X_1

Here $X_0 = b_0c_0 + b_1c_1$, $X_1 = b_0c_1 + b_1c_0$.

The constants to be fitted by the method of least squares are chosen according to the following scheme:

Blocks: $-b_1, b_1; \;\; -b_2, b_2; \;\; \ldots; \;\; -b_q, b_q.$

Interaction BC: $X_0, X_1; \;\; f, -f.$

Interaction ABC: $X_0 a_j, \;\; X_1 a_j; \;\; i_j, \;\; -i_j \, (j = 0, 1, 2, \ldots, q-1),$

$$\sum_{j=0}^{q-1} i_j = 0.$$

Denoting the block totals in the j-th replication by B_{j1}, B_{j2} ($j = 1, 2, \ldots, q$), we take, following Yates,

$$g_j = B_{j1} - B_{j2} \qquad (j = 1, 2, \ldots, q).$$

We also use $[BC]$ to denote, as Yates has done, the ordinary total for this interaction. Similarly, we use $[BC.a_j]$ ($j = 0, 1, 2, \ldots, q-1$) for the total for the interaction in the presence of a_j, the contrast between these q totals giving the interaction ABC.

The normal equations for determining the above constants then come out as under:

$$4q^2 f + 4(q-2)\left(\sum_{i=1}^{q} b_i\right) = [BC]$$

$$4qi_1 + 4qf + 4(-b_1 + b_2 + \ldots + bq) = [BC.a_0]$$

$$4qi_2 + 4qf + 4(b_1 - b_2 + \ldots + bq) = [BC.a_1]$$

....

$$4qi_q + 4\,qf + 4(b_1 + b_2 + \ldots - b_q) = [BC.a_{q-1}]$$

$$4qb_1 + 4(q-2)f + 4(-i_1 + i_2 + \ldots + i_q) = -g_1$$

$$4qb_2 + 4(q-2)f + 4(i_1 - i_2 + \ldots + i_q) = -g_2$$

....

$$4qb_q + 4(q-2)f + 4(i_1 + i_2 + \ldots - i_q) = -g_q$$

By solving the above equations, we obtain

$$16q(q-1)f = q\,[BC] + (q-2)\sum_{i=1}^{q} g_i\,,$$

Taking

$$q\,[BC] + (q-2)\sum_{i=1}^{q} g_i = qQ,$$

we have

$$f = \frac{Q}{16(q-1)}\,.$$

The estimate of BC in units of the yield of a single plot is given by

$$2f = \frac{(qQ)}{8q(q-1)}.$$

The error variance of BC is given by

$$V(2f) = \frac{\sigma^2}{4(q-1)}.$$

In an unconfounded experiment, the estimate of BC would be $(1/2q^2)$ $[BC]$ and its error variance would be σ^2/q^2.

The relative information is, therefore, given by the ratio

$$\frac{1}{q^2} \Big/ \frac{1}{4(q-1)} = \frac{4(q-1)}{q^2},$$

so that the relative loss of information on BC is given by

$$L(BC) = \frac{(q-2)^2}{q^2}.$$

The sum of squares for BC is

$$\frac{Q^2}{16(q-1)} = \frac{(qQ)^2}{16q^2(q-1)}$$

as compared with $(1/4q^2)$ $[BC]^2$ in an unconfounded experiment.

The estimate of ABC is obtained in a similar manner by solution of the normal equations given above. Thus, for estimating i_j $(j = 0, 1, 2, \ldots, q-1)$, we get

$$4(q^2-4)\,i_j + 16(q-1)f = qR_j,$$

where

$$qR_j = q\,[BC.a_j] + g_1 + g_2 + \ldots + g_j - g_{j+1} + g_{j+2} + \ldots + g_q.$$

We thus obtain

$$i_j = \frac{q}{4(q^2-4)}(R_j - \bar{R}) \quad (j = 0, 1, 2, \ldots, q-1)$$

where

$$\bar{R} = \frac{\sum_{j=0}^{q-1} R_j}{q}.$$

The estimate of the interaction ABC, in units of a single plot yield, is given by

$$2i_j\,(j = 0, 1, 2, \ldots, q-1),$$

so that we may write

$$ABC = \frac{q}{2(q^2-4)}\, dev\ (R_0, R_1, \ldots, R_{q-1})$$

$$= \frac{1}{2(q^2-4)}\ dev\ (qR_0, qR_1, \ldots, qR_{q-1})$$

as compared with $(1/2q)\,[BC.a_0]$, etc., in an unconfounded experiment. The error variance applicable to each of these quantities is

$$V(2_{ij}) = \frac{4q\sigma^2}{4(q^2-4)} = \frac{q}{q^2-4}\,\sigma^2$$

as compared with σ^2/q^2 when there is no confounding.

The relative information is, therefore,

$$\frac{q^2-4}{q^2},$$

so that the loss of information on each degree of freedom of ABC is given by

$$L(ABC) = \frac{4}{q^2}.$$

Hence the total loss of information on both the interactions is

$$\frac{(q-2)^2 + 4(q-1)}{q^2} = 1,$$

so that the design is balanced. The sum of squares for the interaction ABC is given by

$$\frac{q}{4(q^2-4)}\, dev^2\,(R_0, R_1, \ldots, R_{q-1})$$

$$= \frac{1}{4q(q^2-4)}\, dev^2\,(qR_0, qR_1, \ldots, qR_{q-1}).$$

8. Summary

The method of finite geometries developed earlier by Bose and Kishen for solving the problem of confounding in the general symmetrical factorial design has been extended to the construction of balanced confounded asymmetrical factorial designs which were not so far amenable to this approach. This has been achieved by using

curvilinear spaces or hypersurfaces and truncating the $EG(m, s)$ suitably. A more general method, using vectors in Galois fields, has also been introduced and a unified theory for the construction of both symmetrical and asymmetrical factorial designs developed. It has been shown that, with the help of this theory, symmetrical confounded factorial designs s^m, where s is not a prime number or a prime power, as also almost all types of asymmetrical factorial designs can be constructed in an optimum manner. Methods of deriving symmetrical and asymmetrical factorial designs, using the b.i.b. property, have also been given, besides methods of reducing the number of replications required for balancing in asymmetrical designs and of deriving balanced designs of the type $a_1s_1 \times a_2s_2 \times \ldots \times a_ms_m$ from a given $s_1 \times s_2 \times \ldots \times s_m$ design. Finally, two methods of analysis of balanced partially confounded designs have been briefly discussed to enable the interested reader to work out formulæ for analysis of any specific design.

9. References

1. Bose, R. C. and Kishen, K. "On the problem of confounding in the general symmetrical factorial design," *Sankhya*, 1940, **5**, 21–36.
2. Kempthorne, O. .. *The Design and Analysis of Experiments*, John Wiley and Sons, London, 1952, p. 359.
3. Kishen, K. .. "Recent developments in experimental design," *Presidential Address to the Section of Statistics of the 45th Indian Science Congress*, Madras, 1958.
4. —— and Srivastava, J. N. "Confounding in asymmetrical factorial designs in relation to finite geometries," *Curr. Sci.*, 1959, **28**, 98–100.
5. Li, Jerome C. R. "Design and statistical analysis of some confounded factorial experiments," *Research Bulletin* 333, Iowa State College of Agriculture, 1944.
6. Nair, K. R. and Rao, C. R. "Confounded designs for asymmetrical factorial experiments," *Sci. & Cult.*, 1941, **7**, 361–62.
7. ———— "A general class of quasifactorial designs leading to confounded designs for factorial experiments," *Ibid.*, 1942, **7**, 457–58.
8. ———— "Confounding in asymmetrical factorial experiments," *J. Roy. Stat. Soc.*, 1948, **10 B**, 109–31.
9. Thompson, H. R. and Dick, I. D. "Factorial designs in small blocks derived from orthogonal Latin Squares," *Ibid.*, 1951, **13 B**, 126–30.
10. Yates, F. .. "The design and analysis of factorial experiments," *Imperial Bureau of Soil Science, Technical Communication* No. 35, 1937.